

KAIST CHEP 95/17
October 1995

General Static Solutions of 2-dimensional Einstein-Dilaton-Maxwell-Scalar Theories

DAHL PARK

Department of Physics

KAIST

Taejon 305-701, KOREA

AND

YOUNGJAI KIEM

Department of Physics

Sejong University

Seoul 133-747, KOREA

E-mail: dpark@chep6.kaist.ac.kr

ykiem@phy.sejong.ac.kr

ABSTRACT

General static solutions of effectively 2-dimensional Einstein-Dilaton-Maxwell-Scalar theories are obtained. Our model action includes a class of 2-d dilaton gravity theories coupled with a $U(1)$ gauge field and a massless scalar field. Therefore it also describes the spherically symmetric reduction of d -dimensional Einstein-Scalar-Maxwell theories. The properties of the analytic solutions are briefly discussed.

1 Introduction

The model action we consider in this paper is given as follows.

$$I = \int d^2x \sqrt{-g} e^{-2\phi} [R + \gamma g^{\alpha\beta} \partial_\alpha \phi \partial_\beta \phi + \mu e^{2\lambda\phi} - \frac{1}{2} e^{-2\phi(\delta-1)} g^{\alpha\beta} \partial_\alpha f \partial_\beta f + \frac{1}{4} e^{\epsilon\phi} F^2], \quad (1)$$

where R denotes the 2-d scalar curvature and F , the curvature 2-form for an Abelian gauge field. ϕ and f represent a dilaton field and a massless scalar field, respectively. The parameters $\gamma, \mu, \lambda, \epsilon$ and δ are assumed to be arbitrary real numbers. A specific choice of these 5 parameters corresponds to a particular gravity theory. Action (1) is of interest in itself as a 2-d dilaton gravity theory coupled with a $U(1)$ gauge field and a scalar field, since it contains various couplings of the dilaton field to other fields [1][2]. In particular, the choice of $\gamma = 4$ and $\lambda = 0$ reduces the gravity sector of the action to the theory of Callan, Giddings, Harvey and Strominger (CGHS) [3]. This string-inspired model has provided us with an analytically tractable framework to study the gravitational physics [2][4]. Additionally, our model represents the spherically symmetric reduction of large class of d -dimensional Einstein-Maxwell-Scalar theories [5]. From this point of view, 2-d dilaton field ϕ is directly related to the geometric radius of each $(d-2)$ -dimensional sphere in d -dimensional spherically symmetric geometry. To be specific, we can write the spherically symmetric d -dimensional metric $g_{\alpha\beta}^{(d)}$ as the sum of longitudinal part and transversal angular part

$$ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta - \exp\left(-\frac{4}{d-2}\phi\right) d\Omega^2$$

where $d\Omega^2$ is the metric of a sphere S^{d-2} with unit radius and we use $(+ - \cdots -)$ metric signature. The spherically symmetric reduction of d -dimensional Einstein-Maxwell-Scalar action

$$I = \int d^d x \sqrt{g^{(d)}} \left(R - \frac{1}{2} g^{(d)\alpha\beta} \partial_\alpha f \partial_\beta f + \frac{1}{4} F^2 \right) \quad (2)$$

becomes Eq.(1) with $\gamma = 4(d-3)/(d-2)$, $\lambda = 2/(d-2)$, $\delta = 1$, $\epsilon = 0$ and μ , a constant depending on the area of $(d-2)$ -dimensional sphere after $(d-2)$ -dimensional angular integration. If $d = 4$, for example, we have $\mu = -2$. The spherically symmetric reduction of 4-dimensiona Einstein-Maxwell-Fermion theories, after the bosonization of the Callan-Rubakov modes of the fermions, also becomes (1) with the above values of the parameters, except $\delta = 0$ in this case [6].

Ongoing debates on the quantum evaporation of black holes [4] provide a major motivation for studying the gravity theories described by the action (1). In this

regard, it has been suggested that the possible end points of the black hole evaporation process are the quantum deformations of the extremal solutions similar to the extremal Reissner-Nordström space-time [7]. A class of 2-dimensional dilaton theories of the type (1), where we have a $U(1)$ gauge field, are useful frameworks for investigating this possibility [7][8][9]. A key issue along this line of investigation is the inclusion of the (quantum) gravitational back reactions.

An important step toward the understanding of those gravity theories, therefore, is to obtain their general classical solutions with exact treatment of the classical gravitational back reactions caused by the $U(1)$ gauge field and the scalar matter field. Even when we are interested in static solutions, this task is largely hampered by the non-linearity of the classical equations of motion. What we then need is a systematic procedure to solve this set of non-linear coupled equations.¹ For some simpler theories of interest, the situation is more favorable. In case of 4-dimensional Einstein-Scalar theory, the methods of Buchdahl [12] and Janis et.al. [13] are reported in literature. More recently, Myers and Perry presented an extensive study of d -dimensional Einstein-Maxwell theory [14].

In Ref.[15], in the absence of $U(1)$ gauge field, we utilized the static remnants of underlying classical conformal invariance to reduce the set of second-order coupled ordinary differential equations (ODEs) to the set of first-order ODEs. The ODEs were summarized as the conservation of the corresponding Noether charges. In the present work, we generalize that approach by including a $U(1)$ gauge field. From this point of view, the main novelty of our work is its method of derivation. In what follows, we construct 4 Noether charges so that the 4 fields, namely, the conformal factor of the metric in conformal gauge, the dilaton field, the scalar field and the $U(1)$ gauge field, can be solved from a set of first-order ODEs. This construction is possible for a general choice of 5 parameter values introduced above. Moreover, under the restriction of $2 - \lambda - \gamma/2 + \epsilon/2 = 0$ and $\delta = 1$, we can go further and get the general static solutions in a closed form. It is regrettable that this restriction is necessary for some technical reasons when we try to obtain the solutions in a closed form. However, the spherically symmetric reduction of d -dimensional Einstein-Maxwell-Scalar theory and the CGHS model, two most important cases in our consideration, satisfy the restriction. Finally, we discuss the properties of our solutions and some aspects of classical back reactions implied by our solutions.

Recently, Gürses and Sermutlu derived general static spherically symmetric so-

¹See [10], [11] and references cited therein for other approaches to this problem

lutions to d -dimensional Einstein-Dilaton-Maxwell theory (with additional dilaton coupling to $U(1)$ gauge field that we do not consider in this paper) [11]. By integrating out the angular dependence from the outset to derive effectively 2-dimensional action, we circumvent some of the technical complexity of their approach while getting the general static solutions for the theories continuously connecting the 4-d s -wave Einstein theory to the CGHS model. Our approach also provides a method that works under the choice of more conventional conformal gauge. Still, it remains to be seen whether the analysis in our fashion can be used in deriving general static solutions in the presence of the coupling between the dilaton field and $U(1)$ gauge field in d -dimensional theories ($d > 2$), as dictated from the low energy limit of string theory [16].

2 The Derivation of General Static Solutions

We start by deriving the static equations of motion. Then the existence of symmetries is pointed out to explicitly construct the corresponding Noether charges. Under the aforementioned restriction, we can further integrate the ODEs to get closed form solutions.

2.1 Static Equations of Motion

The equations of motion are obtained from the action by varying it with respect to the metric tensor, the dilaton field, the massless scalar field and the gauge fields;

$$D_\alpha D_\beta \Omega - g_{\alpha\beta} D \cdot D \Omega + \frac{\gamma}{8} \left[g_{\alpha\beta} \frac{(D\Omega)^2}{\Omega} - 2 \frac{D_\alpha \Omega D_\beta \Omega}{\Omega} \right] + \frac{\mu}{2} g_{\alpha\beta} \Omega^{1-\lambda} \quad (3)$$

$$+ \frac{1}{8} (g_{\alpha\beta} F^2 - 4 g^{\mu\nu} F_{\alpha\mu} F_{\beta\nu}) \Omega^{1-\epsilon/2} - \frac{1}{4} g_{\alpha\beta} \Omega^\delta (Df)^2 + \frac{1}{2} \Omega^\delta D_\alpha f D_\beta f = 0,$$

$$R + \frac{\gamma}{4} \left[\frac{(D\Omega)^2}{\Omega^2} - 2 \frac{D \cdot D \Omega}{\Omega} \right] + (1-\lambda) \mu \Omega^{-\lambda} + \frac{1}{4} (1 - \frac{\epsilon}{2}) \Omega^{-\epsilon/2} F^2 - \frac{\delta}{2} \Omega^{\delta-1} (Df)^2 = 0, \quad (4)$$

$$\delta \Omega^{\delta-1} D \Omega \cdot D f + \Omega^\delta D \cdot D f = 0, \quad (5)$$

$$g^{\mu\alpha} g^{\nu\beta} [(1 - \epsilon/2) D_\nu \Omega (D_\alpha A_\beta - D_\beta A_\alpha) + \Omega (D_\nu D_\alpha A_\beta - D_\nu D_\beta A_\alpha)] = 0, \quad (6)$$

where $\Omega = e^{-2\phi}$ and D denotes the covariant derivative. We choose to work in a conformal gauge as $g_{+-} = -e^{2\rho+\gamma\phi/2}/2$ and $g_{--} = g_{++} = 0$, partly to simplify

our analysis. Moreover, when it comes to quantum aspects of 2-d dilaton gravity reported in literature, the conformal gauge choice has been usual[4]. We also require the negative signature for space-like coordinates and the positive signature for a time-like coordinate. Under this gauge choice, our original action, modulo total derivative terms, can be written as

$$I = \int dx^+ dx^- (4\Omega \partial_+ \partial_- \rho + \frac{\mu}{2} e^{2\rho} \Omega^{1-\lambda-\gamma/4} + \Omega^\delta \partial_+ f \partial_- f - e^{-2\rho} \Omega^{(\gamma-2\epsilon)/4+1} F_{-+}^2) \quad (7)$$

where $F_{-+} = \partial_- A_+ - \partial_+ A_-$. The ϕ is included deliberately in the conformal factor to cancel the kinetic term for the dilaton field up to a total derivative term, thereby helping the task of finding relevant symmetries.

The equations of motion under the conformal gauge are

$$\partial_+ \partial_- \Omega + \frac{\mu}{4} e^{2\rho} \Omega^{1-\lambda-\gamma/4} + \frac{1}{2} e^{-2\rho} \Omega^{(\gamma-2\epsilon)/4+1} F_{-+}^2 = 0, \quad (8)$$

$$\begin{aligned} \partial_+ \partial_- \rho + \frac{\mu}{8} (1 - \lambda - \gamma/4) \frac{e^{2\rho}}{\Omega^{\lambda+\gamma/4}} + \frac{\delta}{4} \Omega^{\delta-1} \partial_+ f \partial_- f \\ - \frac{1}{4} ((\gamma - 2\epsilon)/4 + 1) e^{-2\rho} \Omega^{(\gamma-2\epsilon)/4} F_{-+}^2 = 0, \end{aligned} \quad (9)$$

along with the equations for massless scalar field,

$$\delta(\partial_+ \Omega \partial_- f + \partial_- \Omega \partial_+ f) + 2\Omega \partial_+ \partial_- f = 0 \quad (10)$$

and for gauge fields,

$$\partial_- (e^{-2\rho} \Omega^{(\gamma-2\epsilon)/4+1} F_{-+}) = 0, \quad (11)$$

$$\partial_+ (e^{-2\rho} \Omega^{(\gamma-2\epsilon)/4+1} F_{-+}) = 0. \quad (12)$$

The equations for the abelian gauge field can be solved to give

$$F_{-+} = e^{+2\rho} \Omega^{(-\gamma+2\epsilon)/4-1} Q \quad (13)$$

where Q is a constant. In addition to the equations of motion, we have to impose gauge constraints resulting from our choice of the conformal gauge. They are given by

$$\frac{\delta I}{\delta g^{\pm\pm}} = 0, \quad (14)$$

where I is the original action Eq.(1). From the equations of motion for the metric tensor Eq.(3), we obtain the gauge constraints

$$\partial_\pm^2 \Omega - 2\partial_\pm \rho \partial_\pm \Omega + \frac{1}{2} \Omega^\delta (\partial_\pm f)^2 = 0. \quad (15)$$

Now we have to find the static solutions in terms of equations of motion Eq.(8)-(12) with the gauge constraints Eq.(15). The general static solutions can be found by assuming all functions except the gauge field depend on a single space-like coordinate $x = x^+x^-$. Then from Eq.(13), we observe that the variable F_{-+} , originally defined as $\partial_- A_+ - \partial_+ A_-$, automatically becomes dependent only on x and we can consistently reduce the partial differential equations into the coupled second order ODEs. The resulting ODEs are

$$x\ddot{\Omega} + \dot{\Omega} + \frac{\mu}{4}e^{2\rho}\Omega^{1-\lambda-\gamma/4} + \frac{1}{2}e^{-2\rho}\Omega^{(\gamma-2\epsilon)/4+1}F_{-+}^2 = 0, \quad (16)$$

$$x\ddot{\rho} + \dot{\rho} + \frac{\mu}{8}(1-\lambda-\gamma/4)\frac{e^{2\rho}}{\Omega^{\lambda+\gamma/4}} + \frac{\delta}{4}\Omega^{\delta-1}x\dot{f}^2 \quad (17)$$

$$-\frac{1}{4}((\gamma-2\epsilon)/4+1)e^{-2\rho}\Omega^{(\gamma-2\epsilon)/4}F_{-+}^2 = 0, \quad (18)$$

$$\Omega x\ddot{f} + \Omega\dot{f} + \delta\dot{\Omega}x\dot{f} = 0,$$

and

$$\frac{d}{dx} \left(e^{-2\rho}\Omega^{(\gamma-2\epsilon)/4+1}F_{-+} \right) = 0, \quad (19)$$

where the dot represents taking a derivative with respect to x . The gauge constraints become

$$\ddot{\Omega} - 2\dot{\rho}\dot{\Omega} + \frac{1}{2}\Omega^\delta\dot{f}^2 = 0. \quad (20)$$

The general solutions of the above ODEs are the same as the general static solutions of the original action under a particular choice of the conformal coordinates. The above ODE's except the gauge constraint can also be derived from an action

$$I = \int dx \left[x\dot{\Omega}\dot{\rho} - \frac{\mu}{8}e^{2\rho}\Omega^{1-\lambda-\gamma/4} - \frac{1}{4}\Omega^\delta x\dot{f}^2 + \frac{1}{4}e^{-2\rho}\Omega^{(\gamma-2\epsilon)/4+1}\dot{A}^2 \right. \quad (21)$$

$$\left. - \frac{1}{2}\Omega^{3-\lambda-\epsilon/2}x(\dot{A} - F_{-+})^2 \right]$$

by varying this action with respect to Ω , ρ , f , A and F_{-+} . The field A is introduced to get the ODE for F_{-+} , Eq.(19). In fact, for ρ and Ω , we get the equations with F_{-+} replaced by \dot{A} and additional terms containing $(\dot{A} - F_{-+})^2$. From the equation of motion for F_{-+}

$$\Omega^{3-\lambda-\epsilon/2}x(-\dot{A} + F_{-+}) = 0, \quad (22)$$

we find $\dot{A} = F_{-+}$ to eventually get the same equations for ρ and Ω as before. The ODE for F_{-+} , Eq.(19), can be derived by the equation of motion for A .

2.2 Symmetries, Noether Charges and Explicit Solutions

We observe that the action Eq.(21) has four continuous symmetries

- (a) $f \rightarrow f + \alpha$,
- (b) $A \rightarrow A + \alpha$,
- (c) $x \rightarrow xe^\alpha$, $\rho \rightarrow \rho - \frac{1}{2}\alpha$, $F_{-+} \rightarrow F_{-+}e^{-\alpha}$,
- (d) $x \rightarrow x^{1+\alpha}$, $\rho \rightarrow \rho - \frac{1}{2}(2 - \lambda - \frac{\gamma}{4})\ln(\alpha + 1) - \frac{\alpha}{2}\ln x$, $\Omega \rightarrow \Omega(1 + \alpha)$,
 $f \rightarrow f(1 + \alpha)^{(1-\delta)/2}$, $A \rightarrow A(1 + \alpha)^{(\epsilon+2\lambda-4)/4}$, $F_{-+} \rightarrow F_{-+}x^{-\alpha}(1 + \alpha)^{(2\lambda+\epsilon-8)/4}$.

Here α represents an arbitrary real parameter of each transformation. The symmetries (a) and (b) are clear since field f and A appear only through their derivative. The symmetry (c) is clearly a static remnant of the underlying classical conformal invariance. The only non-trivial symmetry of our problem is (d). This transformation, that changes the action by a total derivative, is in fact a conformal coordinate transformation from x^\pm to tortoise coordinates $\log x^\pm$ followed by some overall scale transformation of relevant fields. The Noether charges for these symmetries are straightforwardly constructed as

$$\begin{aligned}
f_0 &= x\Omega^\delta \dot{f} \\
Q &= e^{-2\rho}\Omega^{(\gamma-2\epsilon)/4+1}\dot{A} - 2\Omega^{3-\lambda-\epsilon/2}x(\dot{A} - F_{-+}) \\
c_0 &= \frac{1}{2}x\dot{\Omega} + x^2\dot{\rho}\dot{\Omega} - \frac{1}{4}\Omega^\delta x^2 \dot{f}^2 + \frac{\mu}{8}xe^{2\rho}\Omega^{1-\lambda-\gamma/4} + \frac{1}{4}e^{-2\rho}\Omega^{(\gamma-2\epsilon)/4+1}x\dot{A}^2 \\
&\quad - \frac{1}{2}\Omega^{3-\lambda-\epsilon/2}x^2(\dot{A}^2 - F_{-+}^2) \\
s &= -c_0 \ln x - \frac{1}{2}x\dot{\Omega}(2 - \lambda - \frac{\gamma}{4}) + x\dot{\rho}\Omega + \frac{\delta-1}{4}\Omega^\delta x \dot{f} f \\
&\quad + \frac{\epsilon+2\lambda-4}{8}e^{-2\rho}\Omega^{(\gamma-2\epsilon)/4+1}\dot{A}A + \frac{1}{2}\Omega - \frac{1}{4}(\epsilon+2\lambda-4)\Omega^{3-\lambda-\epsilon/2}xA(\dot{A} - F_{-+}),
\end{aligned}$$

respectively. The imposition of the gauge constraint Eq.(20) yields $c_0 = 0$. Furthermore, imposing Eq.(22), we simplify the Noether charges as

$$f_0 = x\Omega^\delta \dot{f} \tag{23}$$

$$Q = e^{-2\rho}\Omega^{(\gamma-2\epsilon)/4+1}\dot{A} \tag{24}$$

$$0 = x^2\dot{\rho}\dot{\Omega} + \frac{1}{2}x\dot{\Omega} - \frac{1}{4}\Omega^\delta x^2 \dot{f}^2 + \frac{\mu}{8}xe^{2\rho}\Omega^{1-\lambda-\gamma/4} + \frac{1}{4}e^{-2\rho}\Omega^{(\gamma-2\epsilon)/4+1}x\dot{A}^2 \tag{25}$$

$$s = -\frac{1}{2}x\dot{\Omega}(2 - \lambda - \frac{\gamma}{4}) + x\dot{\rho}\Omega + \frac{\delta-1}{4}\Omega^\delta x \dot{f} f$$

$$+\frac{\epsilon+2\lambda-4}{8}e^{-2\rho}\Omega^{(\gamma-2\epsilon)/4+1}\dot{A}A+\frac{1}{2}\Omega \quad (26)$$

We can rewrite the equations of motion Eq.(16)-(19) in a form that represents the conservation of these Noether charges f_0 , Q , c_0 and s ($c_0 = 0$ by the gauge constraint). When we integrate Eqs.(23)-(26) further to get closed form solutions, we get four additional constants of integration. Among these 7 parameters, the meaning of f_0 and Q , the scalar charge and the $U(1)$ charge, respectively, are clear from the asymptotic behavior of the scalar field and the $U(1)$ gauge field. Not all of the remaining 5 parameters are physically important. We note that adding constant terms to f and A is trivial. See, for example, Eq.(22). Additionally, as we will explain later, two parameters actually represent the degree of freedom in the choice of coordinate systems, namely, the reference time choice and the scale choice. These considerations show that the general static solutions are parameterized by three parameters (including Q and f_0), modulo coordinate transformations.

We can explicitly demonstrate the structure of the solution space by assuming $\delta = 1$ and $2 - \lambda - \gamma/2 + \epsilon/2 = 0$. Moreover, in our further consideration, we only consider the case when $Q > 0$.² By this assumption, we exclude the case when A field becomes degenerate, being a strict constant. By letting $\rho = \bar{\rho} + (2 - \lambda - \gamma/4)(\ln \Omega)/2 - (\ln x)/2$, we get

$$(s - \frac{\epsilon+2\lambda-4}{8}QA)\dot{A} = Qe^{2\bar{\rho}}\Omega^{2-\lambda-\gamma/2+\epsilon/2}\dot{\bar{\rho}} \quad (27)$$

from Eq.(24) and Eq.(26). Here we see the role of the assumption $2 - \lambda - \gamma/2 + \epsilon/2 = 0$. For example, in the spherically symmetric reduction of d -dimensional Einstein gravity, we have $\epsilon + 2\lambda - 4 = -4(d-3)/(d-2)$ and $2 - \lambda - \gamma/2 + \epsilon/2 = 0$. Under this condition, the above equation can be integrated directly to yield

$$2sA + \frac{1}{2}(1+q)QA^2 + c = Qe^{2\bar{\rho}} \quad (28)$$

where $q+1 = -(\epsilon+2\lambda-4)/4$ and c is the constant of integration. From Eq.(23),(24) and (28), we can determine f via,

$$\dot{f} = \frac{f_0}{2sA + (1+q)QA^2/2 + c}\dot{A} \quad (29)$$

² If the $U(1)$ charge vanishes, the whole situation becomes identical to that of Ref.[15], where we already have a complete analysis. The results for $Q < 0$ can be trivially obtained from our results for $Q > 0$.

which, upon integration, becomes

$$f = \frac{f_0}{\sqrt{4s^2 - 2(1+q)Qc}} \ln \left| \frac{A - A_-}{A + A_+} \right| + f_1 \quad (30)$$

where $A_{\pm} = (\sqrt{4s^2 - 2(1+q)Qc} \pm 2s) / [(1+q)Q]$ and f_1 is the constant of integration. The constant of integration f_1 represents the trivial constant term we can add to the scalar field f . Using Eq.(23),(24) and (28), we can rewrite Eq.(25) as

$$0 = 4(1+q) \left(\frac{d\phi}{dA} \right)^2 - 2 \frac{P'(A)}{P(A)} \frac{d\phi}{dA} - \frac{f_0^2}{2P^2(A)} + \frac{2Q + \mu e^{-4(1+q)\phi}/Q}{4P(A)} \quad (31)$$

where $P(A) = (1+q)QA^2/2 + 2sA + c$. The prime indicates the differentiation with respect to A . By differentiating the above equation with respect to A , we have

$$0 = \left[\frac{P'}{P} - 4(1+q) \frac{d\phi}{dA} \right] \left[\frac{d^2\phi}{dA^2} + 2(1+q) \left(\frac{d\phi}{dA} \right)^2 - \frac{f_0^2}{4P^2} \right]. \quad (32)$$

We see that two cases are possible. When the first factor of the above equation is zero, we get,

$$|P(A)| = k\Omega^{-2(1+q)} \quad (33)$$

where k is the constant of integration which is greater than zero. We must verify whether the above result is the true solution of Eq.(31). By substituting the above result into Eq.(31) we get

$$0 = D^2 - \frac{(1+q)\mu}{Q} \frac{P}{|P|} k. \quad (34)$$

where $D^2 = 4s^2 - 2(1+q)Qc + 2(1+q)f_0^2$. So we must fix the constant of integration as

$$k = \frac{D^2 Q}{\mu(1+q)} \frac{|P|}{P} > 0 \quad (35)$$

We have found one solution

$$\Omega^{2(1+q)} = \frac{D^2 Q}{\mu(1+q)} \frac{1}{P}. \quad (36)$$

In the second case where the second factor is zero, we can find the solutions using the result derived in Appendix. For $D^2 = 4s^2 - 2(1+q)Qc + 2(1+q)f_0^2 \neq 0$ we have

$$\phi = \frac{1}{4(1+q)} \left[\ln |P| + 2 \ln |c_1 e^{DI} - 2(1+q)| - DI \right] + c_2 \quad (37)$$

where c_1 and c_2 are constants of integration and $I(A) = \int P(A)^{-1} dA$. Since the original equation Eq.(31) is the first order equations we must fix one of the constants in Eq.(37). By plugging the above equation into Eq.(31) we get

$$e^{-4(1+q)c_2} = \frac{8D^2 Q c_1}{-\mu} \frac{|P|}{P} > 0, \quad (38)$$

and thus,

$$\Omega^{2(1+q)} = \frac{8D^2 Q c_1}{-\mu} \frac{e^{DI}}{P(c_1 e^{DI} - 2(1+q))^2} \quad . \quad (39)$$

For $D^2 = 0$ we have

$$\phi = \frac{1}{4(1+q)} \ln |P(I + c_1)^2| + c_2 \quad (40)$$

By substituting the above equation into Eq.(31) we get

$$e^{-4(1+q)c_2} = -\frac{4Q}{\mu(1+q)} \frac{|P|}{P} > 0 \quad (41)$$

and

$$\Omega^{2(1+q)} = -\frac{4Q}{\mu(1+q)} \frac{1}{P(I + c_1)^2}. \quad (42)$$

Since Q does not vanish, we can find A as a function of x by plugging Eq.(28) into Eq.(24)

$$\ln |x/x_0| = \int \frac{\Omega(A)}{P(A)} dA \quad (43)$$

where x_0 is the constant of integration. Thus, x_0 simply represents the choice of the reference time. The following is a summary table for solutions.

	$D \neq 0$	$D = 0$
solutions	$\frac{D^2 Q}{\mu(1+q)} \frac{1}{P}$	$\frac{-4Q}{\mu(1+q)} \frac{1}{P(I+c_1)^2}$
$\Omega^{2(1+q)}$	$\frac{8D^2 Q c_1}{-\mu} \frac{e^{DI}}{P(c_1 e^{DI} - 2(1+q))^2}$	

Table for Solutions

$$\begin{aligned} D^2 &= 4s^2 - 2(1+q)Qc + 2(1+q)f_0^2 \\ P(A) &= (1+q)QA^2/2 + 2sA + c \\ I(A) &= \int P(A)^{-1} dA \end{aligned}$$

2.3 Properties of Static Solutions

The general solution space consists of some discrete number of 7 dimensional spaces. Among these parameters, the meaning of f_0 and Q are clear since they are the scalar charge and the $U(1)$ charge. Additionally, x_0 simply denotes the degree of freedom in choosing a reference time and f_1 is a trivial addition of a constant term to the scalar field. This leaves us with three parameter space (c, c_1, s) . To get the physical degrees of freedom, we further notice that the transformation $A \rightarrow A + k$ does not produce any physically distinguishable changes in the solutions. The orbit of this transformation ($s \rightarrow s + (1+q)Qk/2$ and $c \rightarrow c - 2sk - (1+q)Qk^2/2$ as can be seen from Eq.(28)) should be modded out from the remaining three dimensional space. Similarly, the scale transformation (corresponding to an arbitrary choice of scale and related to the symmetry (c) in section 2.2) should also be modded out. Thus, the physical solution space is parameterized by (f_0, Q) and a parameter that parameterizes the coset of (c, c_1, s) modulo two transformations above. This additional parameter corresponds to the physical mass of a generic black hole.

Given our general solutions, it is physically very interesting to see what happens when a generic charged black hole tries to carry a scalar hair. The Reissner-Nordström type solutions are found in Eq.(39). The range of A is determined to be $A > A_-$ or $A < -A_+$, because $P(A)$, which is equal to $(1+q)Q(A+A_+)(A-A_-)/2$, should be greater than zero as can be seen in Eq.(28). Since we have $\mu < 0$ for most cases of physical interests, we have $c_1 > 0$ from Eq.(39). When the scalar charge f_0 vanishes, Eq.(39) becomes

$$\Omega^{2(1+q)} = \frac{2D^2(k^2 - 1)}{-\mu(1+q)^2} \left[A + \frac{2s}{(1+q)Q} - \frac{Dk}{(1+q)Q} \right]^{-2} \quad (44)$$

where $k = [c_1 + 2(1+q)]/[c_1 - 2(1+q)]$, which satisfies $|k| > 1$. Since Ω is proportional to some power of the geometric radius of transversal sphere, we require it to vary from 0 to infinity. This gives us further restriction on the range of A as $A_- < A < (-2s + Dk)/[(1+q)Q]$ if we assume $D > 0$ and $k > 1$. The metric becomes

$$g_{+-} = -\frac{P(A)}{2Qx\Omega^{\epsilon/2}} \quad (45)$$

where

$$P(A) = \frac{D^2(k^2 - 1)}{2(1+q)Q} \left[1 - \frac{2}{\Omega^{1+q}} \frac{k}{\sqrt{k^2 - 1}} Q(-\mu/2)^{-1/2} + \frac{Q^2(-\mu/2)}{\Omega^{2(1+q)}} \right]. \quad (46)$$

Note that $P(A_-) = 0$, which shows A_- is the outer horizon. It is straightforward to verify that, for 4-dimensional spherically symmetric case, our metric becomes that of the Reissner-Nordström.

If f_0 dose not vanish, the range of A is determined by Eq.(39) where

$$e^{DI} = \left| \frac{A - A_-}{A + A_+} \right|^{D/\sqrt{4s^2 - 2(1+q)Qc}}. \quad (47)$$

Here we take the same assumption as $f_0 = 0$ case. Since the exponent of Eq.(47) is greater than 1, we have

$$\lim_{A \rightarrow A_-^+} \Omega = 0,$$

which was a finite value in the Reissner-Nordström case ($f_0 = 0$ case) representing the radius of the outer horizon. The range of A is $A_- < A < A_\infty$ where

$$A_\infty = \frac{1}{(1+q)Q} \left[\sqrt{4s^2 - 2(1+q)Qc} \frac{1 + \left(\frac{k-1}{k+1}\right) \sqrt{4s^2 - 2(1+q)Qc/D}}{1 - \left(\frac{k-1}{k+1}\right) \sqrt{4s^2 - 2(1+q)Qc/D}} - 2s \right] \quad (48)$$

and

$$\lim_{A \rightarrow A_\infty^-} \Omega = \infty.$$

The value A_∞ , that corresponds to the spatial infinity, becomes the corresponding value of the Reissner-Nordström type solutions if we set $f_0 = 0$. Now let us see what happened to the outer horizon. From Eq.(39), Eq.(24) and Eq.(28), we get

$$2(1+q) \frac{d\Omega}{dy} = (1+q)Q \left[\frac{D - 2s}{(1+q)Q} - A \right] + \frac{2Dc_1 e^{DI}}{2(1+q) - c_1 e^{DI}} \quad (49)$$

where $y = \ln x$, an asymptotically flat coordinate near the spatial infinity. Since e^{DI} , Eq.(47), is a monotonically increasing function of A in $A_- < A < A_\infty$, we have

$$\lim_{A \rightarrow A_-^+} \frac{d\Omega}{dy} = \begin{cases} 0, & f_0 = 0 \\ \text{positive value,} & f_0 \neq 0 \end{cases} \quad (50)$$

$$\lim_{A \rightarrow A_\infty^-} \frac{d\Omega}{dy} = \infty. \quad (51)$$

The result for $A \rightarrow A_-^+$ when $f_0 = 0$ is as expected since it just shows A_- corresponds to the outer (apparent) horizon. When $f_0 \neq 0$, however, the apparent horizon

does not exist between A_- and A_∞ , the physical range of A . In fact, this is non-trivial to verify, for if we define $F(A)$ as the right hand side of Eq.(49), we have $(dF/dA)|_{A=A_-} < 0$. Thus, it may seem possible to have $2(1+q)\frac{d\Omega}{dy} = F(A) = 0$ between A_- and A_∞ . To investigate this possibility, we calculate the minimum value of $F(A)$. At the minimum point $A = A_0$, we have

$$\left. \frac{4D(1+q)}{2(1+q) - c_1 e^{DI}} \right|_{A=A_0} = D + \sqrt{D^2 + 2(1+q)QP(A_0)}, \quad (52)$$

which follows from the condition $(dF/dA)|_{A=A_0} = 0$. The corresponding minimum value of $F(A)$ is given by

$$2(1+q)\frac{d\Omega}{dy} \Big|_{A=A_0} = \sqrt{2(1+q)f_0^2 + [2s + (1+q)QA_0]^2} - 2s - (1+q)QA_0 \quad (53)$$

which is greater than zero if $f_0 \neq 0$. Therefore, we have $d\Omega/dy > 0$ in the specified physical range of A , which in turn implies there is no apparent horizon in the same range of A . Note that at $A = A_-$, where $\Omega = 0$, the scalar field, Eq.(30), diverges logarithmically. This shows that the would-be horizon is shielded by a naked singularity produced by the diverging scalar field (if $f_0 \neq 0$), just like the case of electrically neutral black holes [15]. This consideration holds in all (not just 4-d Reissner-Nordström) model theories we consider and, thus, illustrates “no-scalar-hair property” [17].

It is clear that our solutions are defined on a local coordinate patch. In the process of getting general static solutions, we found many solutions (branches) that show markedly different behavior from the space-time geometry with asymptotically flat Minkowskian geometry. It will be an interesting exercise to glue them together to construct non-trivial and physically interesting global space-time structures.

Acknowledgments

The earlier version of this work was completed while Y.K. was at Physics Department of Princeton University. Y.K. wishes to thank H. Verlinde for useful discussions. D. Park wishes to thank H.C. Kim for useful discussions.

Appendix

In this section we will find the general solutions of the following second order nonlinear ordinary equations

$$\frac{d^2y}{dx^2} - h \left(\frac{dy}{dx} \right)^2 = \frac{g}{(ax^2 + bx + c)^2} \quad (54)$$

where a, b, c and g are constants, and h is a nonzero constant. Let $z(x) = y' \equiv dy/dx$ and $P(x) = ax^2 + bx + c$. Since $P(x)$ is a second order polynomial, we can guess one simple solution of $z(x)$ as $z_0(x) = (c_1x + c_2)/P(x)$. In fact, if we let $z_0(x) = -(P' + D)/(2hP)$, we find

$$z'_0 - h z_0^2 = \frac{b^2 - 4ac - D^2}{4h} \frac{1}{P^2(x)}. \quad (55)$$

The equality $D^2 = b^2 - 4ac - 4hg$ confirms that $z_0(x)$ is a possible solution. To find the general solutions, let $z(x) = z_0(x) + v(x)$. Then we get the equation for $v(x)$

$$v' - h v^2 - 2h z_0(x) v = 0. \quad (56)$$

By letting $u(x) = P(x)v(x)$, we get

$$\frac{u'}{hu^2 - Du} = \frac{1}{P(x)}, \quad (57)$$

which can be integrated easily.

$D \neq 0$ case

In this case, we get

$$z(x) = -\frac{1}{2h} \left\{ \frac{P'(x)}{P(x)} - \frac{D [h \mp c_1 e^{DI(x)}]}{P(x) [h \pm c_1 e^{DI(x)}]} \right\} \quad (58)$$

for $z(x)$ where $I(x) = \int P(x)^{-1} dx$ and c_1 is a non-negative constant. After the integration over x , we finally get

$$y(x) = -\frac{1}{2h} \left[\ln |P(x)| + 2 \ln |h \pm c_1 e^{DI(x)}| - DI(x) \right] + c_2. \quad (59)$$

where c_2 is a constant of integration. The minus sign in front of c_1 can be absorbed into c_1 so that c_1 can be made less than zero.

$D = 0$ case

In this case, we simply get

$$y(x) = -\frac{1}{2h} \ln \left| P(x) [I(x) + c_1]^2 \right| + c_2 \quad (60)$$

where c_1 and c_2 are constants. Using an elementary method, $I(x)$ is calculated to be

$$I(x) = \int \frac{dx}{ax^2 + bx + c} = \frac{1}{\sqrt{b^2 - 4ac}} \ln \left| \frac{\sqrt{b^2 - 4ac} - b - 2ax}{\sqrt{b^2 - 4ac} + b + 2ax} \right| \quad (61)$$

for $b^2 - 4ac > 0$.

References

- [1] C. Teitelboim, in *Quantum Theory of Gravity*, S. M. Christensen, ed. (Adam Hilger, Bristol, 1984).
- [2] T. Banks and O'Loughlin, Nucl. Phys. **B362**, 649 (1991); R. Mann, Phys. Rev. **D47**, 4438 (1993).
- [3] C.G. Callan, S.B. Giddings, Harvey and A. Strominger, Phys. Rev. **D45**, R1005 (1992).
- [4] S. Giddings, lectures at the Trieste Summer School on High Energy Physics, July 1994, hep-th/9412138; A. Strominger, presented at the 1994 Les Houches Summer School, hep-th/9407100.
- [5] Y. Kiem, Phys. Lett. **B322**, 323 (1994); D. Lowe, Phys. Rev. **D47**, 2446 (1993).
- [6] S.P. Trivedi, Phys. Rev. **D47**, 4233 (1993).
- [7] T. Banks and M. O'Loughlin, Phys. Rev. **D48**, 698 (1993).
- [8] D.A. Lowe and M. O'Loughlin, Phys. Rev. **D48**, 3735 (1993).
- [9] A. Strominger and S.P. Trivedi, Phys. Rev. **D48**, 5778 (1993).
- [10] M. Rakhmanov, Phys. Rev. **D50**, 5155 (1994).
- [11] M. Gürses and E. Sermutlu, to appear in Classical and Quantum Gravity, hep-th/9509076.

- [12] H. A. Buchdahl, Phys. Rev. **115**, 1325 (1959).
- [13] A.I. Janis, D.C. Robinson, J. Winicour, Phys. Rev. **186**, 1729 (1969).
- [14] R.C. Myers and M.J. Perry, Ann. Phys. (N.Y.) **172**, 304 (1986).
- [15] Y. Kiem and D. Park, Princeton Preprint PUPT-1493, hep-th/9504021 to appear in Phys. Rev. **D**.
- [16] G. Horowitz, in String Theory and Quantum Gravity '92, Proceedings of the Trieste Spring School and Workshop, J. Harvey et. al., ed., (World Scientific, 1993) and references cited therein.
- [17] J. Chase, Commun. Math. Phys **19**, 276 (1970); R.H. Price, Phys. Rev. **D5**, 2419 (1972).